DEFINITION. We say that A is an $\langle l_1, l_2 \rangle$ *totally invertible matrix* if every k-by-k submatrix of A with $l_1 \leqslant k \leqslant l_2$ is totally invertible.

We also have

DEFINITION. We say that A is a *precisely* $\langle r, l_1 \leqslant l_2 \rangle$ *totally invertible matrix* if every set of r elements with $l_1 \leqslant r \leqslant l_2$ made of $S(A)$ is Carlian, but there exist subsets of $S(A)$ of cardinality r with $r < l_1$ and $r > l_2$ which are not Carlian sets.

The problem is then to construct matrices A of arbitrarily large magnitude which satisfy the definitions given above.

REMARK. Even though the three constructions of totally invertible matrices discussed in this paper are different in their nature, a "common denominator" can be found. Intuitively one may say that in the three constructions given above a large Carlian set is one in which *every element plays a unique role which is different than the role played by any other element in the construction of a totally invertible matrix*.

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Quasimonotonic Seminorms on $\mathbf{C}^n$ and Their Absolute Mappings

by MOSHE GOLDBERG[9]

As usual, a function

$$S : \mathbf{C}^n \to \mathbf{R}$$

is called a *seminorm* on the space of complex n-tuples if for all $x \in \mathbf{C}^n$ and $\lambda \in \mathbf{C}$

$$S(x) \geqslant 0,$$
$$S(\lambda x) = |\lambda| S(x),$$
$$S(x+y) \leqslant S(x) + S(y).$$

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We call S a *norm* if in addition,

$$S(x) > 0 \qquad \forall x \neq 0.$$

Finally, we call a seminorm S *proper* if $S \neq 0$, but $S(x) = 0$ for some $x \neq 0$.

Given a seminorm S on $\mathbf{C}^n$, we discuss in this note the absolute mapping of S, defined by

$$S^+(x) = S(x^+), \qquad x \in \mathbf{C}^n,$$

where

$$x^+ = \{|\xi_j|\} \in \mathbf{R}^n$$

is the vector obtained by taking the absolute values of the components of $x = \{\xi_j\}$.

Part of the motivation for studying S^+ lies in the fact that it is often considerably easier to compute the values of S^+ than those of S. For example, this is so in the case of the numerical radius [5], which takes on $\mathbf{C}_{n \times n}$, the algebra of $n \times n$ matrices, the form

$$r(A) = \max\{|x^*Ax| : x \in \mathbf{C}^n,\ x^*x = 1\}. \tag{1}$$

while for arbitrary matrices, computing $r(A)$ is often quite tedious and unsatisfactory, the evaluation of $r^+(A) \equiv r(A^+)$ is much more convenient in view of the results of [6].

As in [3], we call S *quasimonotonic* if

$$x, y \in \mathbf{R}^n \quad \text{with } 0 \leqslant x \leqslant y \quad \text{implies} \quad S(x) \leqslant S(y),$$

where the inequalities $0 \leqslant x \leqslant y$ are construed componentwise. With this definition of quasimonotonicity we can prove:

THEOREM 1 [3]. *Let S be a seminorm on $\mathbf{C}^n$. Then S^+ is a seminorm if and only if S is quasimonotonic.*

If S is a norm, then

$$S^+(x) > 0 \qquad \forall x \neq 0.$$

Hence, Theorem 1 immediately implies:

THEOREM 2 [3]. *Let S be a norm on $\mathbf{C}^n$. Then S^+ is a norm if and only if S is quasimonotonic.*

For instance, it was shown in [2] that the numerical radius r in (1) is quasimonotonic on $\mathbf{C}_{n \times n}$; and since r is a norm, then by Theorem 2, r^+ is also a norm.

Throughout the remainder of this note we shall assume that $\mathbf{C}^n$ has been given a structure of an algebra over $\mathbf{C}$. For example, take $\mathbf{C}^n$ with the Hadamard multiplication

$$xy = \{\xi_j\eta_j\}, \qquad x = \{\xi_j\}\ y = \{\eta_j\} \in \mathbf{C}^n.$$

Another example is $\mathbf{C}_{n\times n}$ with the usual matrix multiplication. Note that in the first example,

$$(xy)^+ = x^+y^+ \qquad \forall x, y \in \mathbf{C}^n, \tag{2}$$

and in the second,

$$(AB)^+ \leqslant A^+B^+ \qquad \forall A, B \in \mathbf{C}_{n\times n}. \tag{3}$$

Now that multiplication is defined in $\mathbf{C}^n$, we call a seminorm *WS submultiplicative* (or simply, *multiplicative*) if

$$S(xy) \leqslant S(x)S(y) \qquad \forall x, y \in \mathbf{C}^n.$$

Further, given a seminorm S and a fixed constant $\mu > 0$, we say that μ is a *multiplicativity factor* for S if

$$S(xy) \leqslant \mu S(x)S(y) \qquad \forall x, y \in \mathbf{C}^n.$$

Evidently, if μ_0 is a multiplicativity factor for S, then so is any μ with $\mu \geqslant \mu_0$. Thus, having a seminorm S, the question is whether S has multiplicativity factors, and if so, whether there is a least (best) one.

It has been shown in [4] that if S is a norm on $\mathbf{C}^n$, then S always has multiplicativity factors, the least (best) one being

$$\mu_{\min} \equiv \mu_{\min}(S) = \max\{S(xy) : x, y \in \mathbf{C}^n,\ S(x) = S(y) = 1\}.$$

If, however, S is a proper seminorm on $\mathbf{C}^n$, then S may fail to have multiplicativity factors, and in particular, proper seminorms on $\mathbf{C}_{n\times n}$ may never have such factors [1, 4].

Concerning these remarks, we can prove the following result that associates multiplicativity factors of S with those of S^+.

THEOREM 3 [3]. *Let S be a quasimonotonic seminorm on $\mathbf{C}^n$ with a multiplicativity factor μ, and suppose that*

$$(xy)^+ \leqslant x^+y^+ \qquad \forall x, y \in \mathbf{C}^n. \tag{4}$$

Then μ is a multiplicativity factor for S^+; hence the infima of the multiplicativity factors for S and S^+ satisfy

$$\mu_{\mathrm{inf}}(S) \geqslant \mu_{\mathrm{inf}}(S^+).$$

We emphasize that (4) holds for all common multiplication rules on $\mathbf{C}^n$, including Hadamard's multiplication on $\mathbf{C}^n$ and the standard matrix multiplication on $\mathbf{C}_{n\times n}$, as indicated in (2) and (3).

We already mentioned that the numerical radius r in (1) is a quasimonotonic norm on $\mathbf{C}_{n\times n}$. Further, it is not difficult to see [2, 4] that the best (least) multiplicativity factors for r and r^+ satisfy

$$\mu_{\mathrm{min}}(r) = \mu_{\mathrm{min}}(r^+) = 4.$$

In view of Theorem 3 this may lead us to conjecture that if S is a quasimonotonic norm on $\mathbf{C}^n$, then the best multiplicativity factors for S and S^+ coincide. As shown in our final example, this conjecture is quite false [3].

Take $\mathbf{C}^n$ with Hadamard's multiplication, and consider the one-parameter family of quasimonotonic norms

$$S_\alpha(x) = \frac{1}{1+\alpha}\left[\sum_j |\xi_j| + \alpha\left|\sum_j \xi_j\right|\right], \qquad \alpha > 0, \quad x = \{\xi_j\} \in \mathbf{C}^n.$$

Clearly,

$$S_\alpha^+(x) = \sum_j |\xi_i|,$$

and it is not hard to see that

$$\frac{\mu_{\mathrm{inf}}(S_\alpha^+)}{\mu_{\mathrm{inf}}(S_\alpha)} \underset{\alpha\to\infty}{\to} 0,$$

showing that the ratio

$$\frac{\mu_{\mathrm{inf}}(S^+)}{\mu_{\mathrm{inf}}(S)}$$

can be arbitrarily small.

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Decomposition of a Monic Matrix Polynomial into a Product of Linear Factors

by I. KRUPNIK[10]

It is proved that a monic matrix polynomial with all its elementary divisors of degree not more than 2 can be decomposed into a product of linear factors. (It is well known that such decomposition is possible in the case when all the elementary divisors are linear.) An example which shows that the number 2 mentioned above cannot be replaced by 3 is also given.

0. *Notation*

The matrix-valued function

$$L(\lambda) = \sum_{k=0}^{n-1} \lambda^k A_k + \lambda^n E \qquad \left(A_k, E \in \mathbf{C}^{m\times m}\right) \tag{1}$$

is said to be a monic matrix polynomial of degree n, of size $m \times m$ (where $\mathbf{C}^{m\times m}$ denotes the set of all complex matrices of size $m \times m$, and E denotes the unit matrix). A number $\lambda_0 \in \mathbf{C}$ is called an eigenvalue of $L(\lambda)$ if the equation

$$L(\lambda_0) g_0 = 0$$

has a nonzero solution g_0. Such a vector g_0 is called an eigenvector of $L(\lambda)$ corresponding to λ_0. Vectors $g_1, \dots, g_{s-1}$ are said to be associated with an eigenvector g_0 if

$$\sum_{k=0}^{j} \frac{1}{k!} L^{(k)}(\lambda_0) g_{j-k} = 0 \qquad (j = 1, \dots, s-1).$$

The number s is called the length of the chain $g_0, \dots, g_{s-1}$. The maximal length of a chain composed of an eigenvector g_0 and vectors associated with it is denoted by

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This work is a part of the author's Ph.D. thesis at Ben-Gurion University, under the supervision of Professor A. Markus. A complete version of the work will soon be published [6].